Arc-length controlling method for non-linear analysis

When top point of equilibrium states curve is met, usual force-controlling incremental algorithm is fault.

Fig. 1 Typical view of equilibrium states curve with top points

The force controlling approach is possible to apply when $0 \leq \lambda < \lambda_1$, where $\lambda_1$ corresponds to top point 1. If $\lambda > \lambda_1$, force controlling iterative process is still nonconvergent.

Arc-length algorithm allows one to pass all branches of equilibrium states without any serious problem. The normal plane method [1,2] is applied. The non-linear algorithm with developed arc-length strategy is presented below.

Input parameters:
- $\lambda_{\text{max}}$ - maximal value of load parameter;
- $D_{\text{max}}$ - maximal value of controlling displacement;
- NoSteps - number of assumed increments;
- NoIter - number of equilibrium iterations;
- tol_F - tolerance for residual vector norm;
- tol_L - tolerance for load parameter.

- Start intialization

$\lambda = 0$
• Loop over load increments: \( n = 0, 1, \ldots \)

\[
i = 0 \\
\mathbf{R}_0 = 0 \\
\Delta \tilde{d}^0 = 0 \\
d_{n+1}^0 = d
\]

Where: \( \mathbf{R}_i = \lambda_{n+1}^i \mathbf{F}_{\text{ext}} - \mathbf{N}(d_{n+1}^i) \) - residual vector, \( \lambda_{n+1}^i \) - current value of load parameter, \( \mathbf{F}_{\text{ext}} \) - external load, \( \mathbf{N}(d_{n+1}^i) \) - vector of internal forces; \( d_{n+1}^i \) - current displacement vector.

• Loop over equilibrium iterations: \( i = 0, 1, 2, \ldots < N_{\text{iter}} \)

if \((i == 0 \_or \_update \_gent \_matrix \_on \_each \_iteration)\)

\[
\mathbf{K} \tau = \mathbf{K} \tau(d_{n+1}^i) \\
\mathbf{K} \tau \Delta \mathbf{d}_L = \mathbf{F}_{\text{ext}} \\
\]

if \((i > 0)\)

\[
\mathbf{R}_i = \lambda_{n+1}^i \mathbf{F}_{\text{ext}} - \mathbf{N}(d_{n+1}^i) \\
\text{Check convergence : } \frac{\|\mathbf{R}_i\|}{\|\mathbf{F}_{\text{ext}}\|} \leq \text{tol}_F \text{ and } \frac{\lambda_{n+1}^i - \lambda_{n+1}^{i-1}}{\lambda_{n+1}^i} \leq \text{tol}_L \Rightarrow \text{break loop over } i \\
\mathbf{K} \tau \Delta \tilde{d}_i = \mathbf{R}_i \Rightarrow \Delta \tilde{d}_i \\
\]

Set \( \Delta \lambda_i \)

Update \( d_{n+1}^{i+1}, \lambda_{n+1}^{i+1} \) to the next iteration

\[
d_{n+1}^{i+1} = d_{n+1}^i + \Delta \tilde{d}_i + \Delta \lambda_i \Delta \mathbf{d}_L \\
\lambda_{n+1}^{i+1} = \lambda_{n+1}^i + \Delta \lambda_i
\]

End loop over \( I \)

if \((\lambda_{n+1}^{i+1} > \lambda_{\text{max}} \_or \_D^* > D_{\text{max}}) \) break loop over \( n \)

\( D^* \) - controlling displacement
The arc-length strategy sets the increment of load parameter on each iteration step. At the start of solution \( n = 0; i = 0 \) it is adopted
\[
\Delta \lambda_0 = \frac{\lambda_{\text{max}}}{\text{NoSteps}}; \quad \Delta S = \Delta \lambda_0 \sqrt{1 + \Delta d^T \Delta d_L},
\]
where \( \Delta S \) is an arc-length increment. At the start step of each iteration \( i = 0; n > 0 \)
\[
\Delta \lambda_0 = \frac{\Delta S}{\sqrt{1 + \Delta d^T \Delta d_L}}
\]
and when \( i > 0 \) normal plane method gives
\[
\Delta \lambda_i = -\frac{\Delta d^T \Delta d^0}{1 + \Delta d^T \Delta d^0}.
\]
The Fig.1 illustrates the normal plane method, when matrix is updated only on each increment.

Let us denote:
\[
\bar{\tau} = \begin{pmatrix} \Delta \lambda_0 \\ \Delta d \end{pmatrix} \quad \text{tangent line vector; } \quad \bar{n} = \begin{pmatrix} \Delta \lambda_i \\ \Delta d \end{pmatrix} \quad \text{normal line vector.}
\]

According to normal plane method, plane, which is normal to tangent line on zero iteration step, defines the constraint for computation of load increment \( \Delta \lambda_i \).
Condition of orthogonality is: $\vec{r} \cdot \vec{n} = 0$ or $\Delta \lambda_i \Delta \lambda_i + \Delta d_i \Delta d_i = 0$, where $\Delta d_i = \Delta \vec{d}_i + \Delta \vec{\lambda}_i \Delta \vec{d}_i$, and $\Delta d_i = \Delta \lambda_i \Delta d_i^i$; $\Delta d_i^i - \Delta d_i$ for zero iteration ($i=0$).

Such condition of orthogonality allows one to define $\Delta \lambda_i$, when $i=1,2,\ldots$.

It is possible to show that determinant of $|K| = 0$, when singular point (limit top point or bifurcation one) is achieved. Indeed, $K \Delta \vec{d}_i = R_i$. When given point of plate load parameter — controlling displacement belongs to equilibrium state curve, $K \Delta \vec{d}_i = 0$ because residual vector $R_i = 0$ (equilibrium conditions are satisfied exactly). Last expression is a homogeneous linear equation set. So, if in some point $|K| = 0$, it means that except for trivial solution $\Delta \vec{d}_i = 0$ exists nontrivial one. So, determinant $|K|$ changes the sign, when we pass the singular point. Developed algorithm controls the changing of sign of $|K|$. If singular point will be passed, appropriate warning informs user that current equilibrium state is unstable.

Following dialog-box appears and allows one to set parameters for arc-length method:

![Nonlinear Analysis Algorithm Options](image)

Fig.3 Parameters of arc-length method
Where: load increment number – NoSteps; maximum iteration number for one increment – NoIter; maximum load factor - $\lambda_{\text{max}}$; node number, degree of freedom – assign node number and direction for controlling displacement; maximum displacement for selected degree of freedom - $D_{\text{max}}$; relative tolerance for residual forces – tol_F; relative tolerance for displacements – tol_L.

Arc-length method is applied for nonlinear pushover analysis and is strongly recommended, when FE model has the nonlinear connections. Following example illustrates the possibilities of arc-length method, which automatically allows us to get so complex equilibrium states curve, caused by degradation branches of non-linear hinge characteristics:

Fig. 4 Example of frame structure, loaded by lateral seismic forces
Fig. 5  Bending moment-rotation characteristic of non-linear hinges.

Fig. 6  Equilibrium states curve. Like-tooth paths are caused by degradation branches of non-linear hinge characteristics.
References