Generalized Annular Couette Flow of a Power-Law Fluid

Rajeev Malik and Uday V. Shenoy*

Department of Chemical Engineering, Indian Institute of Technology, Powai, Bombay 400 076, India

The steady laminar axial flow of a power-law non-Newtonian fluid in the annular space between two long coaxial cylinders, with one of them in motion and an imposed pressure gradient, is studied. The pressure gradient may assist or oppose the drag on the fluid due to the moving cylinder. Expressions for the volume rate of flow are derived analytically for two cases, one in which there is a maximum or minimum in the velocity profile within the annular region of interest and another in which there is not. A quantitative criterion is established to distinguish between these two cases. The solutions allow direct calculation of the volumetric flow rate for all values of the annulus aspect ratio, the power-law index, and the dimensionless pressure gradient parameter.

Introduction

Generalized couette flow involves the imposition of a pressure gradient on a system in which a bounding surface is in motion. The generalized *plane* couette flow problem has been solved for the Newtonian fluid (Schlichting, 1955), the Ellis fluid (Wadhwa, 1966), and the power-law fluid (Skelland, 1967; Flumerfelt et al., 1969). The corresponding flow problem in an *annulus* for a power-law fluid was investigated by Lin and Hsu (1980). However, their solution is not complete as they did not consider the possibility of the pressure gradient *opposing* the drag flow. Furthermore, the volumetric flow rate equations they obtained were in the form of definite integrals necessitating numerical quadrature. Here, the complete solution is presented, and the flow integrals are evaluated analytically to obtain simple algebraic expressions, which are of general importance and of particular practical utility in performing quick calculations.

Problem Formulation

Consider a fluid confined to the space between two long coaxial cylindrical surfaces, as in Figure 1. The cylindrical surface of radius R is stationary, while that of radius κR moves with a constant axial velocity V in the positive z direction. Furthermore, there exists a pressure gradient in the z direction, with the pressures being P_0 and P_L at $z = 0$ and L, respectively. The local velocity in the axial direction is denoted by v_z and depends solely on the radial distance r. The system is assumed to be isothermal, and viscous heating effects are neglected.

For the above one-dimensional flow problem, the equation of motion on considering an incompressible fluid and cylindrical coordinates simplifies to

$$\frac{\mathrm{d}(r\tau_{rz})}{\mathrm{d}r} = \frac{\Delta P}{L}r \tag{1}$$

where $\Delta P = P_0 - P_L$. If ξ denotes r/R, then eq 1 on integration yields the shear-stress distribution as

$$\tau_{rz} = \frac{\Delta PR}{2L}\left(\xi - \frac{\lambda^2}{\xi}\right) \tag{2}$$

Here, λ^2 is a dimensionless constant of integration. If λ is real ($\lambda^2 \geq 0$), then λ mathematically corresponds to a dimensionless zero-shear radius (i.e., $\tau_{rz} = 0$ at $\xi = \lambda$); on the other hand, if λ is imaginary ($\lambda^2 < 0$), then λ^2 may be taken to be merely an integration constant.

With the Ostwald–de Waele power-law model to describe the non-Newtonian viscosity, the rheological equation of state is

$$\tau_{rz} = -m\left|\frac{\mathrm{d}v_z}{\mathrm{d}r}\right|^{n-1}\frac{\mathrm{d}v_z}{\mathrm{d}r} \tag{3}$$

If $\phi = v_z/V$ is the dimensionless velocity, then eqs 2 and 3 may be combined and rewritten in the following dimensionless form:

$$\left|\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right|^{n-1}\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = \Lambda\left(\frac{\lambda^2}{\xi} - \xi\right) \tag{4}$$

Here, $\Lambda = [\Delta PR/(2mL)]\,(R/V)^n$ may be viewed as a dimensionless pressure gradient parameter, which may be positive or negative depending on whether the pressure gradient acts in the same or opposite direction to the motion of the moving cylinder. The above differential equation must be solved subject to the boundary conditions

$$\phi = 0 \quad \text{at} \quad \xi = 1 \tag{5a}$$

$$\phi = 1 \quad \text{at} \quad \xi = \kappa \tag{5b}$$

The velocity distribution obtained on solving may be substituted into the following expression for the dimensionless volumetric flow rate:

$$\Omega = \frac{Q}{2\pi R^2 V} = \int_\kappa^1 \xi\phi\,\mathrm{d}\xi \tag{6}$$

Wadhwa (1966) has presented a similar formulation for an Ellis fluid in generalized annular couette flow but has provided no solutions for the volumetric flow rate.

Solutions for the power-law fluid are presented below for two cases: one in which the shear stress does not change sign within the annular region and another in which it does. The two cases must be separately considered as the sign of the dimensionless velocity gradient needs to be predetermined to solve eq 4.

Case I (Velocity Profile without Maximum or Minimum)

In the case where there is no maximum or minimum in the velocity distribution in the range $\kappa < \xi < 1$, the dimensionless velocity gradient $\mathrm{d}\phi/\mathrm{d}\xi$ is always negative. Hence, eq 4 can be written as

$$\left(-\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)^n = \Lambda\left(\xi - \frac{\lambda^2}{\xi}\right) \tag{7}$$

As the right-hand side of eq 7 is always positive, it may be noted that $\Lambda > 0$ implies $\lambda^2 < \kappa^2$ (case Ia) and $\Lambda < 0$ implies $\lambda^2 > 1$ (case Ib). On integrating eq 7 from ξ to 1 after taking the nth root of both sides and using eq 5a, the dimensionless velocity profile obtained is

$$\phi = \int_\xi^1 [\Lambda(x - \lambda^2/x)]^{1/n}\,\mathrm{d}x \tag{8}$$

The dimensionless volumetric flow rate as defined in eq

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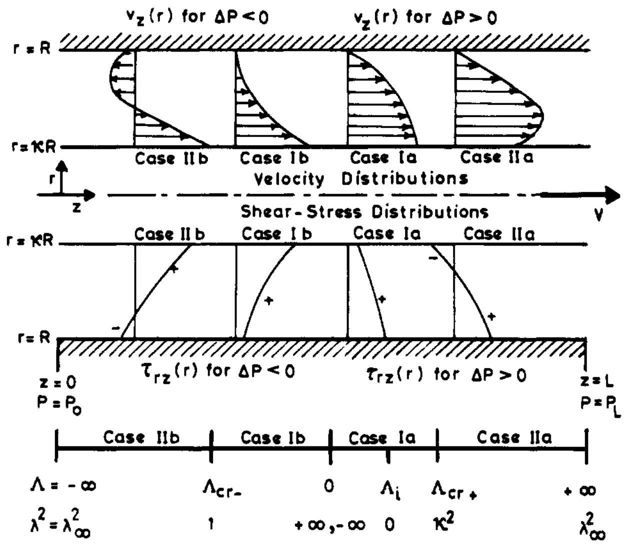

Figure 1. Schematic diagram of generalized annular couette flow.

6 can be now obtained (refer to the Appendix for detailed derivation) as

$$\Omega = \frac{1}{2}\left[\left(\frac{n-1}{3n+1}\right)\lambda^2 - \kappa^2\right] + \frac{n}{2(3n+1)\Lambda}\{[\Lambda(1-\lambda^2)]^{1+1/n} - \kappa^{1-1/n}[\Lambda(\kappa^2-\lambda^2)]^{1+1/n}\} \quad (9)$$

The value of $\lambda(\Lambda,n,\kappa)$ required in eqs 8 and 9 is obtained by imposing the condition in eq 5b on eq 8. Thus

$$\int_\kappa^1 [\Lambda(\xi - \lambda^2/\xi)]^{1/n}\, d\xi = 1 \quad (10)$$

Case II (Velocity Profile with Maximum or Minimum)

In the case where there exists a maximum or minimum in the velocity distribution between the two cylindrical surfaces, $d\phi/d\xi$ changes sign in the range $\kappa < \xi < 1$. Hence, eq 4 must be separately adapted for the two regions $\xi \le \lambda$ and $\xi \ge \lambda$, denoting the respective dimensionless velocity profiles by $\phi^<$ and $\phi^>$:

$$\frac{d\phi^<}{d\xi} = \Lambda|\Lambda|^{1/n-1}\left(\frac{\lambda^2}{\xi} - \xi\right)^{1/n} \quad \text{for} \quad \kappa \le \xi \le \lambda \quad (11a)$$

$$\frac{d\phi^>}{d\xi} = -\Lambda|\Lambda|^{1/n-1}\left(\xi - \frac{\lambda^2}{\xi}\right)^{1/n} \quad \text{for} \quad \lambda \le \xi \le 1 \quad (11b)$$

On integration, the dimensionless velocity profiles obtained by use of the boundary conditions in eq 5 are

$$\phi^< = \Lambda|\Lambda|^{1/n-1}\int_\kappa^\xi (\lambda^2/x - x)^{1/n}\, dx + 1 \quad \text{for} \quad \kappa \le \xi \le \lambda \quad (12a)$$

$$\phi^> = \Lambda|\Lambda|^{1/n-1}\int_\xi^1 (x - \lambda^2/x)^{1/n}\, dx \quad \text{for} \quad \lambda \le \xi \le 1 \quad (12b)$$

The analogue of eq 9 for the dimensionless volume rate of flow in case II can be derived (refer to the Appendix) as

$$\Omega = \frac{1}{2}\left[\left(\frac{n-1}{3n+1}\right)\lambda^2 - \kappa^2\right] + \frac{n\Lambda|\Lambda|^{1/n-1}}{2(3n+1)}\{(1-\lambda^2)^{1+1/n} - \kappa^{1-1/n}(\lambda^2-\kappa^2)^{1+1/n}\} \quad (13)$$

The determining equation for $\lambda(\Lambda,n,\kappa)$ is obtained by equating the velocities given by eqs 12a and 12b at $\xi = \lambda$. Thus

$$\Lambda|\Lambda|^{1/n-1}\left[\int_\kappa^\lambda (\lambda^2/\xi - \xi)^{1/n}\, d\xi - \int_\lambda^1 (\xi - \lambda^2/\xi)^{1/n}\, d\xi\right] = -1 \quad (14)$$

In this case, the physical significance of λ turns out to be the surface ($\xi = \lambda$) where the maximum or minimum in the velocity profile occurs. In other words, the zero-shear surface at $\xi = \lambda$ separates the negative shear-stress region (given by $\kappa \le \xi < \lambda$ for case IIa and by $\lambda < \xi \le 1$ for case IIb) from the positive shear-stress region (given by $\lambda < \xi \le 1$ for case IIa and by $\kappa \le \xi < \lambda$ for case IIb).

Criterion to Distinguish between Cases I and II

It may be noted that case I occurs when the magnitude of the pressure gradient (relative to the velocity V of the moving cylindrical surface) is insufficient to produce a maximum or minimum in the velocity profile within the annular region. Thus, in this case, the maximum velocity occurs at the moving cylindrical surface and the minimum velocity at the stationary one. On the other hand, case II occurs when the magnitude of the pressure is sufficient to produce a maximum (when $\Delta P > 0$) or minimum (when $\Delta P < 0$) in the velocity profile between the two cylindrical surfaces.

Flumerfelt et al. (1969) established a quantitative criterion involving Λ and n to distinguish between these two cases for the problem of generalized *plane* couette flow. Lin and Hsu (1980) claimed that for the problem of generalized *annular* couette flow "such a relation is rather difficult to obtain because of the presence of an additional parameter κ and this does not allow predetermination of what case will result for a particular set of Λ, κ and n". It is demonstrated below that such a quantitative criterion involving Λ, κ, and n is indeed possible. There is no difficulty posed on account of the presence of κ, though it is not always possible to evaluate the integral analytically (except in the Newtonian case and when the reciprocals of n are integers). In either eq 10 or 14 by setting $\lambda = \kappa$ (for $\Lambda > 0$ to obtain Λ_{cr+}) and $\lambda = 1$ (for $\Lambda < 0$ to obtain Λ_{cr-}), the critical values of Λ separating the two cases may be determined. Thus

$$\Lambda_{cr+} = \left[\int_\kappa^1 (\xi - \kappa^2/\xi)^{1/n}\, d\xi\right]^{-n} \quad (15a)$$

$$\Lambda_{cr-} = -\left[\int_\kappa^1 (1/\xi - \xi)^{1/n}\, d\xi\right]^{-n} \quad (15b)$$

The criterion may then be stated as follows: case IIa results for $\Lambda > \Lambda_{cr+}$, case IIb results for $\Lambda < \Lambda_{cr-}$, and case I results for $\Lambda_{cr-} < \Lambda < \Lambda_{cr+}$. Values of Λ_{cr+} and Λ_{cr-} are tabulated in Tables I and II, respectively, for various values of n and κ. The integrals in eq 15 were evaluated numerically by the quadrature routine QDAGS available in IMSL (1987).

Analytical expressions for these critical Λ values may be obtained for the Newtonian fluid by substituting $n = 1$ in eq 15. Thus

$$\Lambda_{cr+} = [0.5(1-\kappa^2) + \kappa^2 \ln \kappa]^{-1} \quad (16a)$$

$$\Lambda_{cr-} = -[0.5(\kappa^2 - 1) - \ln \kappa]^{-1} \quad (16b)$$

Similarly, eq 15 gives for $n = 1/2$

$$\Lambda_{cr+} = [\tfrac{1}{3} - 2\kappa^2 + 8\kappa^3/3 - \kappa^4]^{-1/2} \quad (17a)$$

$$\Lambda_{cr-} = -[-\tfrac{8}{3} + 1/\kappa + 2\kappa - \kappa^3/3]^{-1/2} \quad (17b)$$

Table I. Values of $\Lambda_{cr+}(\kappa,n)$ Computed from Equation 15a

	$\Lambda_{cr+}(\kappa,n)$									
κ	$n = 0.1$	$n = 0.2$	$n = 0.3$	$n = 0.4$	$n = 0.5$	$n = 0.6$	$n = 0.7$	$n = 0.8$	$n = 0.9$	$n = 1.0$
0.05	1.2749	1.4363	1.5597	1.6600	1.7443	1.8170	1.8807	1.9376	1.9888	2.0356
0.10	1.2867	1.4526	1.5812	1.6877	1.7792	1.8600	1.9327	1.9992	2.0609	2.1188
0.20	1.3356	1.5193	1.6682	1.7973	1.9137	2.0214	2.1230	2.2202	2.3142	2.4060
0.30	1.4241	1.6391	1.8223	1.9887	2.1455	2.2967	2.4446	2.5912	2.7376	2.8848
0.40	1.5657	1.8302	2.0671	2.2921	2.5126	2.7328	2.9556	3.1829	3.4165	3.6577
0.50	1.7874	2.1302	2.4529	2.7728	3.0984	3.4348	3.7859	4.1549	4.5446	4.9575
0.60	2.1461	2.6202	3.0901	3.5768	4.0916	4.6428	5.2373	5.8818	6.5829	7.3474
0.70	2.7796	3.5006	4.2567	5.0787	5.9868	6.9991	8.1332	9.4080	10.8442	12.4643
0.80	4.1145	5.4103	6.8651	8.5436	10.5021	12.8000	15.5041	18.6915	22.4519	26.8903
0.90	8.3873	11.8533	16.1546	21.5838	28.4747	37.2380	48.3881	62.5734	80.6128	103.5413
0.95	17.5527	26.6254	38.9343	55.8008	78.9542	110.7266	154.2814	213.9156	295.4676	406.8676

Table II. Values of $\Lambda_{cr-}(\kappa,n)$ Computed from Equation 15b

	$\Lambda_{cr-}(\kappa,n)$									
κ	$n = 0.1$	$n = 0.2$	$n = 0.3$	$n = 0.4$	$n = 0.5$	$n = 0.6$	$n = 0.7$	$n = 0.8$	$n = 0.9$	$n = 1.0$
0.05	-0.0843	-0.1207	-0.1600	-0.2002	-0.2395	-0.2768	-0.3115	-0.3436	-0.3732	-0.4005
0.10	-0.1589	-0.2132	-0.2664	-0.3171	-0.3643	-0.4081	-0.4485	-0.4860	-0.5207	-0.5532
0.20	-0.3082	-0.3918	-0.4685	-0.5393	-0.6052	-0.6668	-0.7251	-0.7806	-0.8339	-0.8854
0.30	-0.4744	-0.5907	-0.6971	-0.7968	-0.8917	-0.9833	-1.0727	-1.1607	-1.2480	-1.3352
0.40	-0.6775	-0.8390	-0.9897	-1.1352	-1.2783	-1.4210	-1.5650	-1.7113	-1.8610	-2.0149
0.50	-0.9478	-1.1788	-1.4017	-1.6245	-1.8516	-2.0862	-2.3306	-2.5870	-2.8573	-3.1432
0.60	-1.3442	-1.6928	-2.0434	-2.4084	-2.7951	-3.2092	-3.6559	-4.1399	-4.6664	-5.2404
0.70	-2.0047	-2.5794	-3.1874	-3.8507	-4.5849	-5.4040	-6.3225	-7.3557	-8.5205	-9.8353
0.80	-3.3535	-4.4688	-5.7275	-7.1835	-8.8852	-10.8846	-13.2400	-16.0192	-19.3012	-23.1784
0.90	-7.6151	-10.8299	-14.8294	-19.8859	-26.3117	-34.4919	-44.9093	-58.1730	-75.0523	-96.5203
0.95	-16.7465	-25.4802	-37.3451	-53.6185	-75.9747	-106.6724	-148.7769	-206.4531	-285.3612	-393.1908

Reduction to Some Specialized Problems

The above expressions include solutions to some specific problems of importance listed below.

(i) Flow of a Power-Law Fluid through a Concentric Annulus under an Imposed Pressure Gradient with Both Cylinders Stationary. This problem of only pressure flow was first studied by Fredrickson and Bird (1958) and is really one extreme of the generalized flow problem considered here. (The other extreme of only drag flow is not directly obtained by reduction of the expressions derived above but can be easily solved independently (Middleman, 1977).) Here, $V = 0$, so eq 13 with $\Lambda \rightarrow \infty$ gives

$$\frac{Q}{\pi R^3} = \frac{n}{(3n+1)}\left(\frac{\Delta PR}{2mL}\right)^{1/n}\{(1-\lambda^2)^{1+1/n} - \kappa^{1-1/n}(\lambda^2-\kappa^2)^{1+1/n}\} \quad (18)$$

The above equation is the result of Hanks and Larsen (1979) and may be used with their table for λ. Their tabulated values of λ for various n and κ are merely the solution to eq 14 when $1/[\Lambda|\Lambda|^{1/n-1}] = 0$; thus, if they are denoted by λ_∞, then the first ($\Lambda \rightarrow -\infty$) and last ($\Lambda \rightarrow \infty$) rows of Tables III–V (included as supplementary material) correspond to λ_∞^2.

(ii) Generalized Annular Couette Flow of a Newtonian Fluid. For $n = 1$, cases I and II become identical, i.e., they need not be considered separately as the absolute sign in eq 4 vanishes. The expression for λ^2 can be determined from either eq 10 or 14 as

$$\lambda^2 = \frac{1}{\ln \kappa}\left[\frac{1}{\Lambda} + \frac{1}{2}(\kappa^2 - 1)\right] \quad \text{where} \quad \Lambda = \frac{\Delta PR^2}{2\mu LV} \quad (19a)$$

Then, the volumetric flow rate equation can be obtained from either eq 9 or 13 as

$$\Omega = \Omega_p + \Omega_d \quad (19b)$$

where $\Omega_p = (\Lambda/8)[(1-\kappa^4) + (1-\kappa^2)^2/(\ln \kappa)]$ and $\Omega_d = (\kappa^2 - 1)/(4 \ln \kappa) - \kappa^2/2$.

Equations 19 are merely another form of the known results of Middleman (1977). For a Newtonian fluid, the solution to the generalized annular couette flow problem is simply the superposition of the solutions to two problems, namely, pressure flow in an annulus (denoted above by subscript p designating the flow between two fixed coaxial cylindrical surfaces due to a pressure gradient only) and drag flow in an annulus (denoted above by subscript d designating the flow between two cylindrical surfaces, one of which is moving, with no pressure gradient). Such a simple superposition is not possible (as can be argued on rigorous mathematical grounds) for non-Newtonian fluids due to the nonlinearity of the shear stress-shear rate relationship.

(iii) Generalized Annular Couette Flow of a Power-Law Fluid with n Being Reciprocal Integers. Formal analytical expressions for ϕ and λ may be obtained from eqs 8, 10, 12, and 14 when the reciprocals of n are integers, along the lines of Fredrickson and Bird (1958). However, these expressions are typically rather cumbersome, and, particularly for the case of λ, the evaluation may not be straightforward. The simplest case in this category (besides the Newtonian) is $n = 0.5$, for which the expressions for λ^2 are given below.

for case I

$$\lambda^2 = \kappa \pm \kappa\left[\frac{2}{3} - \frac{1}{3\kappa} - \frac{\kappa}{3} + \frac{1}{\Lambda^2\kappa(1-\kappa)}\right]^{1/2} \quad (20a)$$

for case II

$$\lambda^4(1 + 1/\kappa) - 16\lambda^3/3 + 2\lambda^2(1+\kappa) - (1+\kappa^3)/3 + 1/(\Lambda|\Lambda|) = 0 \quad (20b)$$

The positive sign in eq 20a is used for $\Lambda < 0$ and the negative sign for $\Lambda > 0$.

(iv) Generalized Plane Couette Flow of a Power-Law Fluid. Since a very thin annulus (with $\kappa = 1 - \epsilon$, where ϵ is small) can be approximated to a thin slit consisting of parallel flat surfaces at $\xi_s = {}^1/_2$ and $\xi_s = -{}^1/_2$, the following transformations hold.

$$\xi = 1 - \epsilon/2 - \epsilon\xi_s \quad (21a)$$

$$\lambda = 1 - \epsilon/2 - \epsilon\lambda_s \quad (21b)$$

$$\Lambda = \Lambda_s/(2\epsilon^{1+n}) \quad (21c)$$

$$\Omega = \Omega_s\epsilon(1 - \epsilon/2) \quad (21d)$$

The subscript s above denotes quantities defined by Flumerfelt et al. (1969) for a thin slit. When these transformations are used in conjunction with any of the previous expressions and the limit as $\epsilon \rightarrow 0$ is evaluated, the results for the generalized plane couette flow (Flumerfelt et al., 1969) are obtained. For example, eqs 15 and 21a give $\Lambda_{cr+} = -\Lambda_{cr-} = (1 + 1/n)^n$ for the planar case. In other words, the last rows in Tables I and II may be approximately obtained from $\Lambda_{cr+} = -\Lambda_{cr-} = (1 + 1/n)^n/(2\epsilon^{1+n})$ with $\epsilon = 0.05$, on considering the $\kappa = 0.95$ case to be a narrow annulus.

Results and Discussion

It must be emphasized that all the above equations are valid for positive Λ (corresponding to cases Ia and IIa where $\Delta P > 0$) as well as negative Λ (corresponding to cases Ib and IIb where $\Delta P < 0$).

Equations 9 and 13 provide complete solutions for the volumetric flow rate during generalized annular couette flow with the value of λ evaluated from eqs 10 and 14 for the respective cases. Such values of λ^2 as a function of Λ and n for three typical values of κ ($\kappa = 0.25, 0.50$, and 0.75) are given in Tables III–V (available as supplementary material). The various cases are indicated in these tables and are separated by horizontal lines. Figure 2 shows a typical plot of λ^2 vs Λ for $\kappa = 0.5$ and $n = 0.2$ with the segments of the curves corresponding to the four cases labeled. As n increases, λ_∞^2 (which is the limiting value of λ^2 when $\Lambda \rightarrow \pm\infty$) increases.

The numerical values in the tables for λ^2 were generated by solving eqs 10 and 14 iteratively using the Newton–Raphson technique. The integrals involved were evaluated numerically by the IMSL (1987) routine QDAGS. The appropriate ranges for both Λ and λ^2 are indicated in Figure 1 for the various cases. These are useful in obtaining suitable initial guesses for the different ranges, while the Newton–Raphson method is employed to solve eqs 10 and 14. It may be remarked that there is a subrange ($0 < \Lambda < \Lambda_i$) in case Ia where λ^2 is negative. The value of Λ_i can be easily determined by setting $\lambda = 0$ in eq 10. Thus

$$\Lambda_i = \left[\frac{1 + 1/n}{1 - \kappa^{1+1/n}}\right]^n \quad (22)$$

MacSporran (1982) has pointed out that this value of Λ_i corresponds to a linear shear-stress distribution.

Tables for λ given in Lin and Hsu (1980) for various values of κ, n, and β ($\beta = 1/\Lambda^{1/n}$ in our notation) do not cover the entire range of possible β values and have meaningless zero entries (except for $n = 1$) as per the comments of MacSporran (1982). Lin and Hsu (1980) limited their discussion to the problem of the pressure gradient assisting the drag flow (i.e., positive values of Λ) and provided tables that were further limited (to the range $\Lambda_i \leq \Lambda < \infty$). Given these deficiencies in the tables of Lin and Hsu (1980), Tables III–V are complete and appropriately revised as suggested by MacSporran (1982). In these tables, the entries for $n = 0.5$ conform with eq 20.

The values of λ^2 may be substituted in eqs 8 and 12 to obtain the velocity profiles. Figure 3 shows typical velocity profiles (in the form of a plot of ξ vs ϕ) for various values of Λ with $\kappa = 0.5$ and $n = 0.2$. The velocity distributions

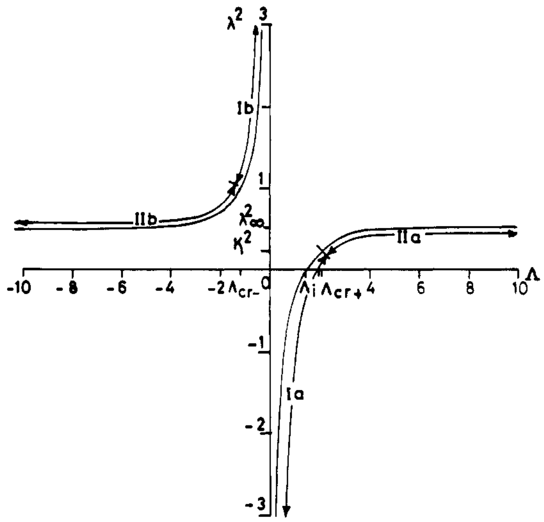

Figure 2. λ^2 as a function of Λ for $\kappa = 0.5$ and $n = 0.2$.

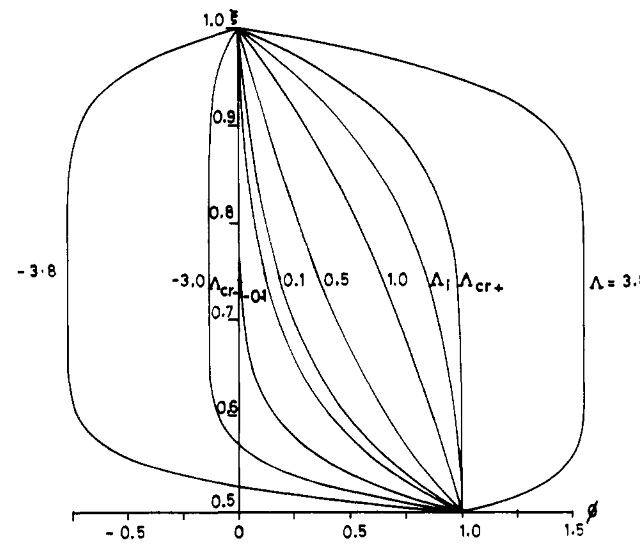

Figure 3. Velocity distributions for different values of Λ with $\kappa = 0.5$ and $n = 0.2$.

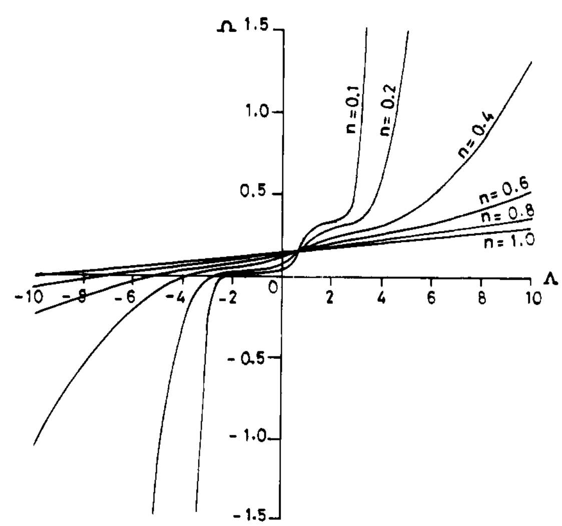

Figure 4. Dimensionless volumetric flow rate as a function of Λ and n for $\kappa = 0.5$.

for Λ_{cr-} (=−1.1788), Λ_i (=1.4355), and Λ_{cr+} (=2.1302) shown in Figure 3 have zero derivatives ($d\phi/d\xi = 0$) at $\xi = 1$, $\xi = 0$, and $\xi = \kappa$, respectively.

Finally, the volumetric flow rate may be simply calculated from eqs 9 and 13 to obtain a dimensionless plot of

Ω vs Λ for a given κ (0.5), as in Figure 4. The relationship is linear in the Newtonian case. The curves exhibit an "apparent" intersection point and two inflections (for low n). Fenner (1970), who also observed such behavior, has provided explanations for these observations.

The solutions presented above would help in analyzing certain lubrication and coating problems, as well as the operations of dies and extruders in polymer processing.

Nomenclature

I_1, I_2 = integrals as defined in eqs A2 and A6, respectively
L = length of cylindrical surfaces
m, n = rheological parameters associated with power-law behavior
P = pressure
ΔP = pressure drop over the length L
Q = volumetric flow rate
R = radius of stationary cylindrical surface
r = radial distance in cylindrical coordinates
s = subscript denoting quantities defined by Flumerfelt et al. (1969) for generalized plane couette flow
V = constant velocity of moving cylindrical surface
v_z = local velocity in z direction
x = dummy variable for integration
z = axial distance in cylindrical coordinates

Greek Letters

ϵ = dimensionless gap for a narrow annulus
κ = ratio of the radius of the moving cylindrical surface to that of the stationary one
Λ = dimensionless pressure gradient parameter as defined in eq 4
Λ_i = value of Λ corresponding to linear shear-stress distribution (as given by eq 22)
Λ_{cr+}, Λ_{cr-} = critical values of Λ separating cases I and II (as given by eq 15)
λ^2 = dimensionless integration constant (λ corresponds to dimensionless zero-shear radius in case II)
λ_∞ = value of λ when $\Lambda \to \pm\infty$
μ = viscosity of Newtonian fluid
ξ = dimensionless distance in radial direction
τ_{rz} = component of stress tensor
ϕ = dimensionless velocity in z direction
$\phi^<$, $\phi^>$ = dimensionless velocity profiles for $\xi \le \lambda$ and $\xi \ge \lambda$ in case II
Ω = dimensionless volumetric flow rate as defined in eq 6

Appendix: Evaluating the Flow Integrals

Integrating eq 6 by parts and using eq 5 give

$$\Omega = -\frac{\kappa^2}{2} - \int_\kappa^1 \frac{\xi^2}{2}\frac{d\phi}{d\xi}d\xi \tag{A1}$$

Case I. Substituting from eq 7 into eq A1 gives

$$\Omega = -\frac{\kappa^2}{2} + \frac{I_1}{2} \quad \text{where} \quad I_1 = \int_\kappa^1 [\Lambda(\xi^2 - \lambda^2)]^{1/n}\xi^{2-1/n}\, d\xi \tag{A2}$$

On utilizing eq 10, I_1 can be expressed as

$$I_1 = \frac{1}{\Lambda}\int_\kappa^1 [\Lambda(\xi^2 - \lambda^2)]^{1+1/n}\xi^{-1/n}\, d\xi + \lambda^2 \tag{A3}$$

Also, on integrating by parts, eq A2 gives

$$I_1 = \frac{n}{2\Lambda(n+1)}\left\{[\Lambda(1-\lambda^2)]^{1/n+1} - \kappa^{1-1/n}[\Lambda(\kappa^2 - \lambda^2)]^{1/n+1} - \frac{n-1}{n}\int_\kappa^1 [\Lambda(\xi^2 - \lambda^2)]^{1/n+1}\xi^{-1/n} d\xi\right\} \tag{A4}$$

Combining eqs A3 and A4 gives

$$I_1 = \frac{n}{\Lambda(3n+1)}\{[\Lambda(1-\lambda^2)]^{1/n+1} - \kappa^{1-1/n}[\Lambda(\kappa^2 - \lambda^2)]^{1/n+1}\} + \frac{n-1}{3n+1}\lambda^2 \tag{A5}$$

On substituting in eq A2, the final expression for Ω given in eq 9 is obtained.

Case II. By an analogous procedure, substituting from eq 11 into eq A1 results in

$$\Omega = -\frac{\kappa^2}{2} + \frac{I_2}{2} \quad \text{where} \quad I_2 = \Lambda|\Lambda|^{1/n-1}\left\{-\int_\kappa^\lambda (\lambda^2 - \xi^2)^{1/n}\xi^{2-1/n}\, d\xi + \int_\lambda^1 (\xi^2 - \lambda^2)^{1/n}\xi^{2-1/n}\, d\xi\right\} \tag{A6}$$

On utilizing eq 14, I_2 can be written as

$$I_2 = \Lambda|\Lambda|^{1/n-1}\int_\kappa^1 |\lambda^2 - \xi^2|^{1+1/n}\xi^{-1/n}\, d\xi + \lambda^2 \tag{A7}$$

Proceeding as before, integrating eq A6 by parts and combining the result with eq A7 give

$$I_2 = \frac{n\Lambda|\Lambda|^{1/n-1}}{3n+1}\{(1-\lambda^2)^{1/n+1} - \kappa^{1-1/n}(\lambda^2 - \kappa^2)^{1/n+1}\} + \frac{n-1}{3n+1}\lambda^2 \tag{A8}$$

With substitution in eq A6, the final expression for Ω given in eq 13 is obtained.

It may be pointed out that the above evaluations of the flow integrals utilize some of the ideas originally proposed by Hanks and Larsen (1979), who obtained analytical expressions for the volumetric flow rate during the flow of a power-law fluid in an annulus with an imposed pressure gradient and both cylinders stationary. The derivation route adopted here is simpler as it does not involve iterated integrals.

Supplementary Material Available: Tables III–V, containing values of λ^2 as a function of Λ and n for κ = 0.25, 0.50, and 0.75 (3 pages). Ordering information is given on any current masthead page.

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